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On conformal β-changes of more generalized m-throot metrics

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ABSTARCT:

A change of Finsler metric $F(x,y) \to \overline{F}(x,y)$ is called a conformal β-change of F, if $\overline{F}(x,y) = e^{\sigma(x)}F(x,y) + \beta(x,y)$, where $\beta(x,y) = b_i(x)y^i$ is a one-form on an n-dimensional smooth manifold M and $\sigma(x)$ is conformal factor. The present paper is devoted mainly to studying the conditions for more generalized m-th root metrics $\widetilde{F}_1 = \sqrt{{A_1}^{\frac{2}{m_1}} + B_1} + C_1$ *and* $\widetilde{F}_2 = \sqrt{{A_2}^{\frac{2}{m_2}} + B_2} + C_2$, When is established conformal β-change.

Keywords: *m-th root metric; more generalized m-th root metric;Randers β-change;conformalβ-change.*

1. Introduction

Studying Finsler geometry one encounters substantial difficulties trying to seek analoguesof classical global, or sometimes even local, results of Riemannian geometry. These difficultiesarise mainly from the fact that in Finsler geometry all geometric objects depend not only onpositional coordinates, as in Riemannian geometry, but also on directional arguments.

Let (M,F) be an n-dimensional Finsler manifold. For a differential one-form $\beta(x,y) = b_i(x)y^i$ on M, G. Randers [1], in 1941, introduced a special Finsler space defined by the change $\overline{F}(x,y) = F(x,y) + \beta(x,y)$, where F is Riemannian. M. Matsumoto [2], in 1974, studied Randers space and generalized Randers space in which F is Finslerian. On the other hand, in 1976, M. Hashiguchi [3] studied the conformal change of Finsler metrics, namely, $\overline{F}(x,y) = e^{\sigma(x)}F(x,y)$. In particular, he also dealt with the special conformal transformation named C-conformal. This change has been studied by many authors ([4], [5]). In 2008, S. Abed ([6], [7]) introduced the transformation

$$\overline{F}(x,y) = e^{\sigma(x)}F(x,y) + \beta(x,y). \tag{1.1}$$

Moreover, he established the relationships between some important tensors associated with (M,F) and the corresponding tensors associated with $(M,\overline{F})$. He also studied some invariant and σ-invariant properties and obtained a relationship between the Cartan connection associated with (M, F) and the transformed Cartan connection associated with $(M, \bar{F})$.

In 1979, Shimada [8] introduced the m-th root metric on the differentiable manifold M defined as:

$$F = \sqrt[m]{a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}} \qquad (1.2)$$

wherethe coefficients $a_{i_1 i_2 \dots i_m}$ are the components of symmetric covariant tensor field of order $(0, m)$being the functions of positional co-ordinates only. Since then various geometers such as [9], [10], etc. have explored the theory of m-th root metric and studied its transformations.

There exist the following important one class of Finsler metric,

$$\bar{F} = \sqrt{A^{\frac{2}{m}} + B} \,,$$

$\tilde{F} = \sqrt{A^{\frac{2}{m}} + B} + C$, (1.3)

where$A = a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$, $B = b_{ij}(x) y^i y^j$ and$C = c_k(x) y^k$, that is one 1-form. This forms are called a generalized m-th root metric and more general generalized m-th root metric, respectively. Obviously, $\tilde{F}$ is not reversible Finsler metric and is Randers change of generalized m-th root metric $\bar{F}$.

In this paper, we have considered a transformation of the more generalized m-th root metricsuch that it transforms to a similar metric as the conformalβ-change one de_ned in (1.1) ina way that the Riemannian metric F is replaced with more generalized m-th root metrics $\tilde{F}$defined in (1.3). Then, we obtain the conditions among two more generalized m-th root metrics $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1and\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$due to conformal β-change.

In overall this paper,

$$A_1 = a_{i_1 i_2 \dots i_{m_1}}(x) y^{i_1} y^{i_2} \dots y^{i_{m_1}} \,, \qquad (1.4)$$

$A_2 = \bar{a}_{i_1 i_2 \dots i_{m_2}}(x) y^{i_1} y^{i_2} \dots y^{i_{m_2}}$,

$B_1 = b_{ij}(x) y^i y^j$,

$$B_2 = \bar{b}_{ij}(x) y^i y^j \,,$$

$$C_1 = c_k(x) y^k,$$

and m_1, m_2 are belongs to natural numbers.

2. Main results

case 1: m_1, m_2 are even numbers and $m_1 = m_2$.

Theorem 2.1 *Let* $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ *and* $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ *are two more generalized m-th root metrics on an open subset* $U \subset R^n$, *where* A_1, B_1, C_1, A_2, B_2 *and* C_2 *are given by (1.4). Suppose that* m_1, m_2 *are even numbers with* $m_1 = m_2$ *and* $B_1 = e^{2\sigma(x)}B_2$. *If* $\tilde{F}_1$ *is conformal* β-change *of* $\tilde{F}_2$, *then* $A_1 = \pm e^{m_1}A_2$ *(or or* $A_1 = \pm e^{m_2}A_2$*) and* $C_1 = e^{\sigma(x)}C_2 + \beta$.

Proof. For simplicity, we put $m_1 = m_2 = m$. Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1} + C_1 = e^{\sigma(x)}\left(\sqrt{A_2^{\frac{2}{m}} + B_2} + C_2\right) + \beta. \tag{2.1}$$

By putting (-y) instead of (y) in (2.1), we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1} - C_1 = e^{\sigma(x)}\left(\sqrt{A_2^{\frac{2}{m}} + B_2} - C_2\right) - \beta. \tag{2.2}$$

Summing sides the two equations (2.1) and (2.2), we have

$$A_1^{\frac{2}{m}} + B_1 = e^{2\sigma(x)}A_2^{\frac{2}{m}} + e^{2\sigma(x)}B_2. \tag{2.3}$$

Consequently, we get the proof.■

We have the following.

Corollary 2.1 *Let* $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ *and* $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ *are two more generalized m-th root metrics on an open subset* $U \subset R^n$ *where* A_1, B_1, C_1, A_2, B_2 *and* C_2 *are given by (1.4). Suppose that* m_1, m_2 *are even numbers with* $m_1 = m_2 = m$ *and* $B_1 = e^{2\sigma(x)}B_2$. *If* $\sqrt[m]{A_1}$ *and* $\sqrt[m]{A_2}$ are Riemannian metrics, then $\tilde{F}_1 = e^{\sigma(x)}\tilde{F}_2$ *if and only if* $A_1 = \pm e^{2\sigma(x)}A_2$ *and* $C_1 = e^{\sigma(x)}C_2$.

case2: m_1, m_2 are odd numbers and $m_1 = m_2$.

Theorem 2.2 *Let* $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ *and* $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ *are two more generalized m-th root metrics on an open subset* $U \subset R^n$, *where* A_1, B_1, C_1, A_2, B_2 *and* C_2 *are given by (1.4). Suppose that* m_1, m_2 *are odd* *numbers with* $m_1 = m_2$*and* $B_1 = e^{2\sigma(x)}B_2$. *If* $\tilde{F}_1$ *is conformal* β-changeof $\tilde{F}_2$, *then* $A_1 = \pm e^{m_1}A_2, A_1 = \pm ie^{m_1}A_2$*(or* $A_1 = \pm e^{m_2}A_2,\ A_1 = \pm ie^{m_2}A_2$ *)and* $C_1 = e^{\sigma(x)}C_2 + \beta$.

Proof.For simplicity, we put $m_1 = m_2 = m$. Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1}+C_1=e^{\sigma(x)}(\sqrt{A_2^{\frac{2}{m}} + B_2}+C_2) + \beta. \qquad (2.4)$$

By putting (-y) instead of (y) in (2.4), we have

$$\sqrt{-A_1^{\frac{2}{m}} + B_1} - C_1=e^{\sigma(x)}(\sqrt{-A_2^{\frac{2}{m}} + B_2} - C_2) - \beta. \qquad (2.5)$$

Summing sides the two equations (2.4) and (2.5), we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1} + \sqrt{-A_1^{\frac{2}{m}} + B_1} = e^{\sigma(x)}(\sqrt{A_2^{\frac{2}{m}} + B_2} + \sqrt{-A_2^{\frac{2}{m}} + B_2}). \qquad (2.6)$$

Thus

$$B_1 + \sqrt{(B_1)^2 - A_1^{\frac{4}{m}}} = e^{2\sigma(x)}(B_2 + \sqrt{(B_2)^2 - A_2^{\frac{4}{m}}}). \qquad (2.7)$$

Therefore,

$$B_1 + \sqrt{(B_1)^2 - A_1^{\frac{4}{m}}} = e^{2\sigma(x)}B_2 + \sqrt{(e^{2\sigma(x)}B_2)^2 - (e^{m\sigma(x)}A_2)^{\frac{4}{m}}} \qquad (2.8)$$

Because of $B_1 = e^{2\sigma(x)}B_2, A_1 = \pm e^{m_1}A_2, A_1 = \pm ie^{m_1}A_2$ and then $C_1 = e^{\sigma(x)}C_2 + \beta$.■

Theorem 2.3*Let* $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1}+C_1$ *and* $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2}+C_2$ *are two more generalized m-th root metrics on an open subset* $U \subset R^n$, *where*A_1,B_1,C_1, A_2,B_2 *and* C_2 *are given by (1.4). Suppose that* m_1, m_2 *are odd numbers with* $m_1 = m_2 = m$*and*$B_1 \neq e^{2\sigma(x)}B_2$. *If* $\tilde{F}_1$ *is conformal*β-changeof $\tilde{F}_2$, *then*$m = 1$.

Proof.From (2.8), we have

$$A_1^{\frac{4}{m}} + (e^{m\sigma(x)}A_2)^{\frac{4}{m}} + 2\sqrt{(e^{2\sigma(x)}B_1B_2)^2 - (e^{2\sigma(x)}B_2)^2A_1^{\frac{4}{m}} - (B_1)^2(e^{m\sigma(x)}A_2)^{\frac{4}{m}} + (e^{m\sigma(x)}A_1A_2)^{\frac{4}{m}}}. \qquad (2.9)$$

By (1.4), one can see that$m = 1$.■

case3: m_1, m_2 are even numbers and $m_1 \neq m_2$.

Theorem 2.4*Let* $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ *and* $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ *are two more generalized m-th root metrics on an open subset* $U \subset R^n$*, where*A_1,B_1,C_1, A_2,B_2 *and* C_2 *are given by (1.4). Suppose that* m_1, m_2 *are even numbers with* $m_1 \neq m_2$, $m_1 > m_2$*and*$B_1 = e^{2\sigma(x)} B_2$. *If* $\tilde{F}_1$ *is conformal* β-change*of*$\tilde{F}_2$, *then* $A_1 = \pm(e^{m_2\sigma(x)} A_2)^{\frac{m_1}{m_2}}$*and* $C_1 = e^{\sigma(x)} C_2 + \beta$.

Proof.Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1 = e^{\sigma(x)}\left(\sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2\right) + \beta. \tag{2.10}$$

By putting (-y) instead of (y) in (2.10), we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} - C_1 = e^{\sigma(x)}\left(\sqrt{A_2^{\frac{2}{m_2}} + B_2} - C_2\right) - \beta. \tag{2.11}$$

Summing sides the two equations (2.10) and (2.11), we have

$$A_1^{\frac{2}{m_1}} + B_1 = e^{2\sigma(x)}\left(A_2^{\frac{2}{m_2}} + B_2\right). \tag{2.12}$$

Consequently, we get the proof.■

In above theorem, if $m_1 - m_2 = k$, where k is even number, then by (2.12), we get

(a_1): If $\frac{k}{m_2} > 1$, then

Case 1:$\frac{k}{m_2} = 2t$. Therefore, from theorem 2.4, $A_1 = \pm(e^{\frac{k}{2t}\sigma(x)} A_2)^{1+2t}$.

Case 2:$\frac{k}{m_2} = 2t + 1$. Therefore, from theorem 2.4, $A_1 = \pm(e^{\frac{k}{2t+1}\sigma(x)} A_2)^{2(1+t)}$.

Case 3: $m_2 \nmid k$. Because of$k = m_2 q + r$, from theorem 2.4, $A_1 = \pm(e^{\frac{k-r}{q}\sigma(x)} A_2)^{1+q+\frac{r}{m_2}}$.

(a_2): If $\frac{k}{m_2} = 1$, then, from theorem 2.4, $A_1 = \pm(e^{k\sigma(x)} A_2)^2$.

(a_3): If $\frac{k}{m_2} < 1$, then

Case 1: $\frac{m_2}{k} = 2t$. Therefore, from theorem 2.4, $A_1 = \pm(e^{2kt\sigma(x)} A_2)^{\frac{1+2t}{2t}}$.

Case 2: $\frac{m_2}{k} = 2t + 1$. Therefore, from theorem 2.4, $A_1 = \pm(e^{(2t+1)k\sigma(x)} A_2)^{\frac{2+2t}{1+2t}}$.

Case 3: $k \nmid m_2$. Because of $m_2 = k\acute{q} + \acute{r}$, from theorem 2.4, $A_1 = \pm(e^{(k\acute{q}+\acute{r})\sigma(x)} A_2)^{1+\frac{k}{k\acute{q}+\acute{r}}}$.

case4: m_1, m_2 are odd numbers and $m_1 \neq m_2$.

Theorem 2.5*Let* $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ *and* $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ *are two more generalized m-th root metrics on an open subset* $U \subset R^n$, *where* A_1, B_1, C_1, A_2, B_2 *and* C_2 *are given by (1.4). Suppose that* m_1, m_2 *are odd numbers with* $m_1 \neq m_2$, $m_1 > m_2$ *and* $B_1 = e^{2\sigma(x)}B_2$. *If* $\tilde{F}_1$ *is conformal* β-change*of* $\tilde{F}_2$, *then* $A_1 = \pm(e^{m_2\sigma(x)}A_2)^{\frac{m_1}{m_2}}$, $A_1 = \pm i(e^{m_2\sigma(x)}A_2)^{\frac{m_1}{m_2}}$ *and* $C_1 = e^{\sigma(x)}C_2 + \beta$.

Proof. Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1 = e^{\sigma(x)}(\sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2) + \beta. \tag{2.13}$$

By putting (-y) instead of (y) in (2.13), we have

$$\sqrt{-A_1^{\frac{2}{m_1}} + B_1} - C_1 = e^{\sigma(x)}(\sqrt{-A_2^{\frac{2}{m_2}} + B_2} - C_2) - \beta. \tag{2.14}$$

Summing sides the two equations (2.13) and (2.14), we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} + \sqrt{-A_1^{\frac{2}{m_1}} + B_1} = e^{\sigma(x)}(\sqrt{A_2^{\frac{2}{m_2}} + B_2} + \sqrt{-A_2^{\frac{2}{m_2}} + B_2}). \tag{2.15}$$

Thus

$$B_1 + \sqrt{(B_1)^2 - A_1^{\frac{4}{m_1}}} = e^{2\sigma(x)}(B_2 + \sqrt{(B_2)^2 - A_2^{\frac{4}{m_2}}}). \tag{2.16}$$

Therefore,

$$B_1 + \sqrt{(B_1)^2 - A_1^{\frac{4}{m_1}}} = e^{2\sigma(x)}B_2 + \sqrt{(e^{2\sigma(x)}B_2)^2 - (e^{m_2\sigma(x)}A_2)^{\frac{4}{m_2}}}. \tag{2.17}$$

Because of $B_1 = e^{2\sigma(x)}B_2$, we get $A_1 = \pm(e^{m_2\sigma(x)}A_2)^{\frac{m_1}{m_2}}$, $A_1 = \pm i(e^{m_2\sigma(x)}A_2)^{\frac{m_1}{m_2}}$ and then $C_1 = e^{\sigma(x)}C_2 + \beta$.■

case5: m_1, m_2 are even and odd numbers, respectively.

Theorem 2.6*Let* $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ *and* $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ *are two more generalized m-th root metrics on an open subset* $U \subset R^n$, *where* A_1, B_1, C_1, A_2, B_2 *and* C_2 *are given by (1.4). Suppose that* m_1, m_2 *are even and odd numbers, respectively and* $B_1 = e^{2\sigma(x)}B_2$. *If* $\tilde{F}_1$ *is conformal* β-change*of* $\tilde{F}_2$, *then* $A_1 = \pm \sqrt[\frac{m_1}{2}]{\frac{1}{2}(-B_1 \pm \sqrt{(B_1)^2 - (e^{m_2\sigma(x)}A_2)^{\frac{4}{m_2}}})}$ *and* $C_1 = e^{\sigma(x)}C_2 + \beta$.

Proof. Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1 = e^{\sigma(x)}(\sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2) + \beta. \tag{2.18}$$

By putting (-y) instead of (y) in (2.18), we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} - C_1 = e^{\sigma(x)}(\sqrt{-A_2^{\frac{2}{m_2}} + B_2} - C_2) - \beta. \tag{2.19}$$

Summing sides the two equations (2.18) and (2.19), we have

$$2\sqrt{A_1^{\frac{2}{m_1}} + B_1} = e^{\sigma(x)}(\sqrt{A_2^{\frac{2}{m_2}} + B_2} + \sqrt{-A_2^{\frac{2}{m_2}} + B_2}). \tag{2.20}$$

Thus

$$4A_1^{\frac{4}{m_1}} + 4(B_1)^2 + e^{4\sigma(x)}(B_2)^2 - 4e^{2\sigma(x)}B_1B_2 + 4A_1^{\frac{2}{m_1}}\left(2B_1 - e^{2\sigma(x)}B_2\right) = \left(e^{2\sigma(x)}B_2\right)^2 - (e^{m_2\sigma(x)}A_2)^{\frac{4}{m_2}}. \tag{2.21}$$

Because of $B_1 = e^{2\sigma(x)}B_2$, we have

$$4A_1^{\frac{4}{m_1}} + 4(B_1)A_1^{\frac{2}{m_1}} + (e^{m_2\sigma(x)}A_2)^{\frac{4}{m_2}} = 0\,. \tag{2.22}$$

Consequently, $A_1 = \pm \sqrt[\frac{m_1}{2}]{\frac{1}{2}(-B_1 \pm \sqrt{(B_1)^2 - (e^{m_2\sigma(x)}A_2)^{\frac{4}{m_2}}})}$ and then $C_1 = e^{\sigma(x)}C_2 + \beta.$ ■

3. Generalized Conformal h-vector-change in Finsler spaces

We investigated what we call a conformal h-vector-change in Finsler spaces, namely

$$F(x,y) \to \bar{F}(x,y) = e^{\sigma(x)}F(x,y) + \beta, \tag{3.1}$$

where, σ is a function of x only, and $\beta(x,y) := b_i(x,y)y^i$, where $b_i \coloneqq b_i(x,y)$ is an h-vector. This change generalizes various types of changes: conformal changes, generalized Randers changes, Randers change. Under this change, we obtain the relationships between some tensors associated with (M,F) and the corresponding tensors associated with $(M,\bar{F})$ [11].

In this paper, we introduce a general transformation or changeof Finsler metrics, which is referred to as a generalized conformal h-vector-change in Finsler spaces:

$$F(x,y) \to \bar{F}(x,y) = f(e^{\sigma(x)}F(x,y) + \beta), \tag{3.2}$$

to investigate the characteristics of this change For those interested. This transformation combines both$h$-vector-change and conformal change in a general setting.Some properties of conformal h-vector-change in Finsler spaces, as fundamental Finsler connections,together with their associated geometric objects, are obtained [11].

The main results of this section are being investigated about it, for publication in journals for later.

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